$$\frac{\mathrm{d}}{\mathrm{d}x}(h\overline{u}) = \underset{\sim}{w}, \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\overset{(\mathrm{i})}{\frac{77}{48}}h\overline{u}^2\right) = -g_\perp h\overset{(\mathrm{ii})}{\frac{\mathrm{d}\tilde{z}}{\mathrm{d}x}} - \overset{(\mathrm{iii})}{\frac{\mu_W}{W}}g_\perp h^2, \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\underset{(\mathrm{iv})}{(\kappa h\overline{u}^3)} = -\underset{(\mathrm{v})}{g_\perp h\overline{u}}\frac{\mathrm{d}\tilde{z}}{\mathrm{d}x} - \frac{5}{9}\underset{(\mathrm{vi})}{\frac{\mu_W}{W}}g_\perp h^2\overline{u} - \frac{35}{9}\underset{(\mathrm{vii})}{\chi D}g_\perp^{1/2}\frac{\overline{u}^2}{h^{1/2}}. \tag{3}$$

* We seek to balance the terms in the momentum & kinetic energy equations. $\left([\xi] = \xi_c\right) \equiv$ the scale of variable ξ

* We seek relationships between:
(i) the scale of the free surface $[\tilde{z}]$ and
(ii) the scale of the flow depth $[h]$

* Accordingly $\left[\frac{d\tilde{z}}{dx}\right] \equiv \left[\frac{\tilde{z}}{\ell}\right] = \frac{[\tilde{z}]}{\ell} = \frac{[\tilde{z}]}{R}$ ... $\ell = R$ for a $\frac{1}{2}$ full drum

$$\therefore \left[\frac{d\tilde{z}}{dx}\right] = \frac{[\tilde{z}]}{R} \rightarrow (1.1)$$

This is equivalent to writing:

$[d\tilde{z}] \equiv [\tilde{z}] \rightarrow (1.2)$

$[dx] \equiv [x] \rightarrow (1.3)$

NB: To use this equivalence, the change for $d\tilde{z}$ or dx must be balance another change. This usually occurs in a derivative

$[(ii)] = [(iii)]$

$$\Rightarrow \cancel{g_\perp}[h]\left[\frac{d\tilde{z}}{dx}\right] = \left(\frac{\mu_w}{L}\right)\cancel{g_\perp}[h]^2$$

From (1.3):

$$\frac{[d(h\bar{u}^2)]}{dx} = \frac{[h\bar{u}^2]}{[x]}$$

$$\Rightarrow \frac{[\tilde{z}]}{R} = \left(\frac{\mu_w}{L}\right)[h] \quad \ldots \text{from } (1.1)$$

$$\Rightarrow \boxed{[\tilde{z}] = \left(\frac{\mu_w}{L}\right)R[h]} \rightarrow (2)$$

similarly: $[(i)] = [(ii)] \Rightarrow \frac{[d(h\bar{u}^2)]}{[dx]} = g_\perp[h]\left[\frac{d\tilde{z}}{dx}\right]$

$$\Rightarrow \frac{\cancel{[h]}[\bar{u}]^2}{[x]} = g_\perp\cancel{[h]}\left[\frac{d\tilde{z}}{dx}\right] \Rightarrow \frac{[\bar{u}]^2}{\cancel{R}} = \frac{g_\perp[\tilde{z}]}{\cancel{R}} \ldots \text{from } (1.1) \rightarrow (*)$$

subst (2) into (*) :

$$\therefore \boxed{[\bar{u}] = \sqrt{g_\perp\left(\frac{\mu_w}{L}\right)R[h]}} \rightarrow (3)$$

$$[(v)] = [(vii)]$$

$$\Rightarrow \quad g_\perp [h][\bar{u}]\left[\frac{dz}{dx}\right] = (\chi D)(g_\perp)^{\frac{1}{2}} \frac{[\bar{u}]^2}{[h]^{\frac{1}{2}}}$$

$$\Rightarrow (g_\perp)^{\frac{1}{2}} [h]^{3/2} \frac{[z]}{R} = (\chi D)[\bar{u}]$$

$$\Rightarrow [\bar{u}] = \frac{(g_\perp)^{\frac{1}{2}}[h]^{3/2}}{(\chi D) R}$$ → now replace $[h]$ using equ. 2

$$= \frac{(g_\perp)^{\frac{1}{2}}[h]^{3/2}}{(\chi D) R} \left(\frac{\mu_w}{L}\right) R\,[h]$$ ... from 2

$$\therefore \boxed{[\bar{u}] = \frac{(g_\perp)^{\frac{1}{2}}}{(\chi D)}\left(\frac{\mu_w}{L}\right)[h]^{5/2}} \longrightarrow 4$$

$$③ = ④$$

$$\Rightarrow \cancel{(g\perp)}^{\frac{1}{2}}\left(\frac{\mu_w}{L}\right)^{\frac{1}{2}} R^{\frac{1}{2}} [h]^{\frac{1}{2}} = \frac{\cancel{(g\perp)}^{\frac{1}{2}}}{(\chi D)}\left(\frac{\mu_w}{L}\right)[h]^{5/2}$$

$$[h]^2 = \frac{R^{\frac{1}{2}}(\chi D)}{\left(\frac{\mu_w}{L}\right)^{1/2}}$$

$$\Rightarrow \boxed{[h] = \frac{R^{\frac{1}{4}}(\chi D)^{\frac{1}{2}}}{\left(\frac{\mu_w}{L}\right)^{1/4}}} \longrightarrow ⑤$$

subst. 5 into 4

$$\Rightarrow [\bar{u}] = \frac{\left(\frac{\mu_w}{L}\right)(g_\perp)^{\frac{1}{2}}}{(\chi D)} \quad \frac{R^{\frac{1}{4}\times\frac{5}{2}} \quad (\chi D)^{\frac{1}{2}\times\frac{5}{2}}}{\left(\frac{\mu_w}{L}\right)^{1/4 \times 5/2}}$$

$$\therefore \boxed{[\bar{u}] = \left(\frac{\mu_w}{L}\right)^{3/8} (g_\perp)^{\frac{1}{2}} R^{5/8} (\chi D)^{\frac{1}{2}}} \longrightarrow 6$$

subst. ⑤ into ②

$$\Rightarrow [\tilde{z}] = \frac{\left(\frac{\mu_w}{L}\right) R R^{\frac{1}{4}} (XD)^{1/2}}{\left(\frac{\mu_w}{L}\right)^{1/4}}$$

$$\Rightarrow \boxed{[\tilde{z}] = \left(\frac{\mu_w}{L}\right)^{3/4} R^{5/4} (XD)^{\frac{1}{2}}} \longrightarrow ⑦$$

Noting that $S = -\frac{d\tilde{z}}{dx} \Rightarrow [S] = \left[\frac{d\tilde{z}}{dx}\right] \equiv [\tilde{z}]/R$ ... from (1.1)

and from ⑦ $\Rightarrow$

$$[S] = \frac{\left(\frac{\mu_w}{L}\right)^{3/4} R^{5/4} (XD)^{\frac{1}{2}}}{R}$$

$$\therefore \boxed{[S] = \left(\frac{\mu_w}{L}\right)^{3/4} (XD)^{\frac{1}{2}} R^{\frac{1}{4}}} \longrightarrow ⑧$$

Characteristic Time scale : $[t]$

$$[t] \triangleq \frac{[x]}{[\bar{u}]} = \frac{R}{[\bar{u}]} \xrightarrow{\text{full def}^n}$$

$$\bar{u}_c \triangleq \frac{dx_c}{dt_c}$$

$$\Rightarrow \bar{u}_c \equiv \frac{x_c}{t_c} = \frac{R}{t_c} \quad \cdots \text{from } 13$$

$$\therefore t_c = \frac{R}{\bar{u}_c}$$

$\cancel{\&}$ substituting equ. 6 for $[\bar{u}]$ we get

$$[t] = \frac{R\left(\frac{L}{\mu_w}\right)^{3/8}}{(g_\perp)^{\frac{1}{2}} R^{5/8} (\chi D)^{1/4}} = \frac{R^{3/8}\left(\frac{L}{\mu_w}\right)^{3/8}}{(g_\perp)^{1/2}(\chi D)^{1/4}}$$

$$\therefore \boxed{[t] = \frac{R^{3/8}\left(\frac{L}{\mu_w}\right)^{3/8}}{(g_\perp)^{\frac{1}{2}}(\chi D)^{1/4}}} \longrightarrow 9$$

Characteristic timescale: An equivalent definition

Capart (2015) defines characteristic timescale $[t]$ as:

$$[t] = [t_i] \times \frac{[h]}{[XD]}$$

$[h]$ ← macroscopic length scale

$[XD]$ ← microscopic length scale

(XD) remains const. for mono-sized grains in a steady-state drum

where $\boxed{[t_i] = \sqrt{\frac{[h]}{g_\perp}}}$

think gravitational free fall from rest thro' the typical depth of the flowing layer:

$$h = \frac{1}{2} g_\perp t_i^2 \quad \therefore \quad t_i = \sqrt{\frac{2h}{g_\perp}} \Rightarrow [t_i] = \sqrt{\frac{[h]}{g_\perp}}$$

NB: $[t]$ is

Timescales continued...

Given the equivalence of the two timescale definitions, this implies:

* time for perturbat^{ns} of size $\ell = R$ to propagate along slope

$$\cong$$

time for perturbat^{ns} of size h_c to propagate orthogonally through the flowing layer

$$\therefore \quad t_c = \frac{(h_c)^{3/2}}{(g_\perp)^{\frac{1}{2}}(\chi D)}$$

$$= \frac{\left\{R^{\frac{1}{4}}(\chi D)^{\frac{1}{2}}\left(\frac{L}{\mu_w}\right)^{\frac{1}{4}}\right\}^{3/2}}{(g_\perp)^{\frac{1}{2}}\ (\chi D)}$$

$$= \frac{R^{3/8}(\chi D)^{3/4}\left(\frac{L}{\mu_w}\right)^{3/8}}{(g_\perp)^{\frac{1}{2}}(\chi D)} = \frac{R^{3/8}\left(\frac{L}{\mu_w}\right)^{3/8}}{(g_\perp)^{1/2}(\chi D)^{1/4}}$$

$$\therefore \quad \boxed{t_c = \frac{R^{3/8}\left(\frac{L}{\mu_w}\right)^{3/8}}{(g_\perp)^{1/2}(\chi D)^{1/4}}}$$

... this is equivalent to equation 9

Let ξ_c be the characteristic scale of a given system variable ξ.
Then noting that equations (5) – (9) are characteristic variables, we have

$$x_c = R$$

$$\tilde{z}_c = \left(\frac{L}{\mu_w}\right)^{-3/4} R^{5/4} (\chi D)^{\frac{1}{2}}$$

$$h_c = R^{\frac{1}{4}} (\chi D)^{\frac{1}{2}} \left(\frac{L}{\mu_w}\right)^{\frac{1}{4}}$$

$$\bar{u}_c = \left(\frac{L}{\mu_w}\right)^{-3/8} (g_\perp)^{\frac{1}{2}} R^{5/8} (\chi D)^{1/4}$$

$$t_c = \frac{R^{3/8} \left(\frac{L}{\mu_w}\right)^{3/8}}{(g_\perp)^{1/2} (\chi D)^{1/4}}$$

$$S_c = \left[\frac{d\tilde{z}}{dx}\right] \equiv \frac{\tilde{z}_c}{x_c} = \left(\frac{L}{\mu_w}\right)^{-3/4} R^{1/4} (\chi D)^{1/2}$$

Free & Forced entrainment

Forced Entrainment: $[f] = \omega x_c$

Free Entrainment: $[e] = \dfrac{h_c}{t_c}$

$\hat{\omega} = \dfrac{f_c}{e_c}$ ... Entrainment #

for uniform unsteady flows

$$\therefore \hat{\omega} = \frac{[f]}{[e]} = \frac{\omega x_c t_c}{h_c} = \frac{\omega R t_c}{h_c} = \frac{\omega R h_c^{3/2}}{(g_\perp)^{1/2}(XD)\, h_c} = \frac{\omega R h_c^{\frac{1}{2}}}{(g_\perp)^{1/2}(XD)}$$

$$\Rightarrow \hat{\omega} = \frac{\omega R}{(g_\perp)^{1/2}(XD)} \frac{R^{\frac{1}{8}}(XD)^{\frac{1}{4}}\left(\frac{L}{\mu_w}\right)^{1/8}}{1}$$

$$\therefore \boxed{\hat{\omega} = \frac{\omega R^{9/8}}{(g_\perp)^{\frac{1}{2}}(XD)^{3/4}}\left(\frac{L}{\mu_w}\right)^{1/8}}$$

... Entrainment # (dimensionless)

Relating Characteristic variable (ε_c) to their dimensionless counterparts ($\hat{\varepsilon}$)

$\hat{\tilde{Z}} = \dfrac{\tilde{Z}}{Z_c}$; $\hat{x} = \dfrac{x}{x_c} = \dfrac{x}{R}$ ... for a $\frac{1}{2}$ filled drum of radius R

$\hat{S} = \dfrac{S}{S_c}$; $\hat{\bar{u}} = \dfrac{\bar{u}}{\bar{u}_c}$; $\hat{h} = \dfrac{h}{h_c}$

Non-dimensionlised system of Equations

$$x = \hat{x}\, x_c = \hat{x} R$$

$$\Rightarrow dx = R\, d\hat{x} \quad \ldots \quad R \equiv \text{const.}$$

$$h = h_c \hat{h} \quad \& \quad \bar{u} = \hat{u}\, \bar{u}_c$$

subst. into (28)

$$\Rightarrow \frac{d}{R\, d\hat{x}}\left[h_c \hat{h}\, \hat{\bar{u}}\, \bar{u}_c\right] = -\omega \hat{x} R$$

$$\Rightarrow \frac{d}{d\hat{x}}\left[R^{\frac{1}{4} \times \frac{2}{2}} (\chi D)^{\frac{1}{2}} \left(\frac{L}{\mu_w}\right)^{\frac{1}{4}} \hat{h} \left(\frac{L}{\mu_w}\right)^{-\frac{3}{8}} (g_\perp)^{\frac{1}{2}} R^{5/8} (\chi D)^{\frac{1}{4}} \hat{u}\right]$$

$$= -\omega \hat{x} R^2$$

$$\Rightarrow \frac{d}{d\hat{x}}\left[\frac{(\chi D)^{3/4}\,(g_{\perp})^{\frac{1}{2}}}{\left(\frac{L}{\mu_w}\right)^{\frac{1}{8}} R^{9/8}}\,\hat{h}\,\hat{\bar{u}}\right] = -\,\omega\,\hat{x}$$

$$\Rightarrow \frac{d}{d\hat{x}}\left[\hat{h}\,\hat{\bar{u}}\right] = -\left\{\frac{\omega R^{9/8}\left(\frac{L}{\mu_w}\right)^{\frac{1}{8}}}{(\chi D)^{3/4}\,(g_{\perp})^{\frac{1}{2}}}\right\}\hat{x}$$

$$\Rightarrow \boxed{\frac{d}{d\hat{x}}\left[\hat{h}\,\hat{\bar{u}}\right] = -\,\hat{\omega}\,\hat{x}} \rightarrow (11)$$

Now let's non-dimensionalise equ. (29) in intermediate regime

$$dx = R\,d\hat{x} \; ; \; h = h_c \hat{h} \; ; \; \bar{u} = \bar{u}_c \hat{u} \; ; \; S = -\frac{dZ}{dx} = S_c \hat{S}$$

$\therefore$ equ. (29) becomes :

$$\frac{d}{R\,d\hat{x}}\left[\gamma h_c \hat{h} \bar{u}_c^2 \hat{u}^2\right] = g_\perp h_c \hat{h} S_c \hat{S} - \left(\frac{\mu_w}{L}\right) g_\perp h_c^2 \hat{h}^2$$

$$\Rightarrow \frac{d}{d\hat{x}}\left[\gamma \hat{h} \hat{u}^2\right] = \frac{R g_\perp \cancel{h_c} \hat{h} S_c \hat{S}}{\cancel{h_c} \bar{u}_c^2} - \frac{\left(\frac{\mu_w}{L}\right) g_\perp h_c^2 \hat{h}^2 R}{h_c \bar{u}_c^2}$$

$$= \frac{R g_\perp \hat{h} \hat{S}}{1} \underbrace{\left(\frac{S_c}{\bar{u}_c^2}\right)}_{B_1} - \left(\frac{\mu_w}{L}\right) \frac{g_\perp R}{1} \hat{h}^2 \underbrace{\left(\frac{h_c}{\bar{u}_c^2}\right)}_{B_2}$$

$$B_1 = \frac{S_c}{U_c^2} = \frac{\cancel{(\chi D)^{1/2}}\, R^{1/4}}{\cancel{\left(\frac{L}{\mu_w}\right)^{3/4}}} \times \frac{\cancel{\left(\frac{L}{\mu_w}\right)^{3/4}}}{g_\perp R^{5/4}\, \cancel{(\chi D)^{1/2}}}$$

$$= \frac{1}{g_\perp R}$$

$$B_2 = \frac{h_c}{U_c^2} = \frac{R^{\frac{1}{4}}\, \cancel{(\chi D)^{\frac{1}{2}}} \left(\frac{L}{\mu_w}\right)^{\frac{1}{4}}}{1} \times \frac{\left(\frac{L}{\mu_w}\right)^{3/4}}{g_\perp R^{5/4}\, \cancel{(\chi D)^{1/2}}}$$

$$= \frac{\left(\frac{L}{\mu_w}\right)}{R\, g_\perp} = \frac{1}{R\, g_\perp \left(\frac{\mu_w}{L}\right)}$$

$$\therefore \frac{d}{d\hat{x}}\left[\gamma \hat{h}\hat{\bar{u}}^2\right] = \left(\frac{Rg_\perp}{Rg_\perp}\right)\hat{h}\hat{S} - \frac{\frac{\mu_w}{L} Rg_\perp}{Rg_\perp\left(\frac{\mu_w}{L}\right)}\hat{h}^2$$

$$\Rightarrow \boxed{\frac{d}{d\hat{x}}\left[\gamma \hat{h}\hat{\bar{u}}^2\right] = \hat{h}\hat{S} - \hat{h}^2} \longrightarrow (12)$$

Now lets non-dimensionalise equ. (30) in intermediate regime

$$dx = R\,d\hat{x} \;;\; h = h_c\hat{h} \;;\; \bar{u} = \bar{u}_c\hat{\bar{u}} \;;\; S = -\frac{dZ}{dx} = S_c\hat{S}$$

∴ equ. (30) becomes:

$$\frac{d}{R\,d\hat{x}}\left[K h_c\hat{h}\bar{u}_c^3\hat{\bar{u}}^3\right] = g_\perp h_c\hat{h}\bar{u}_c\hat{\bar{u}}S_c\hat{S} - \frac{5}{9}\left(\frac{\mu_w}{L}\right)g_\perp h_c^2\hat{h}^2\bar{u}_c\hat{\bar{u}} - \frac{\frac{35}{9}\chi D g_\perp^{\frac{1}{2}}\bar{u}_c^2\hat{\bar{u}}^2}{h_c^{\frac{1}{2}}\hat{h}^{1/2}}$$

$$\frac{d}{d\hat{x}}\left(K\hat{h}\hat{\bar{u}}^3\right)\left(\frac{h_c\bar{u}_c^3}{R}\right) = \hat{h}\hat{\bar{u}}\hat{S}\left(g_\perp h_c\bar{u}_c S_c\right) - \frac{5}{9}\hat{h}^2\hat{\bar{u}}\left\{\left(\frac{\mu_w}{L}\right)g_\perp h_c^2\bar{u}_c\right\} - \frac{35}{9}\frac{\hat{\bar{u}}^2}{\hat{h}^{1/2}}\left[\frac{\chi D g_\perp^{\frac{1}{2}}\bar{u}_c^2}{h_c^{1/2}}\right]$$

× by $\frac{R}{h_c\bar{u}_c^3}$ ⇒

$$\frac{d}{d\hat{x}}\left(K\hat{h}\hat{\bar{u}}^3\right) = \hat{h}\hat{\bar{u}}\hat{S}\left[\frac{R g_\perp \cancel{h_c}\cancel{\bar{u}_c}S_c}{\cancel{h_c}\bar{u}_c^{\cancel{3}\,2}}\right] - \frac{5}{9}\hat{h}^2\hat{\bar{u}}\left\{\left(\frac{\mu_w}{L}\right)\frac{g_\perp h_c^{\cancel{2}\,1}\cancel{\bar{u}_c}R}{\cancel{h_c}\bar{u}_c^{\cancel{3}\,2}}\right\} - \frac{35}{9}\frac{\hat{\bar{u}}^2}{\hat{h}^{1/2}}\left[\frac{R\chi D g_\perp^{\frac{1}{2}}\cancel{\bar{u}_c^2}}{\cancel{h_c^{1/2}}\,h_c^{3/2}\,\bar{u}_c^{\cancel{3}\,1}}\right]$$

$$\frac{d}{d\hat{x}}\left[K\hat{h}\hat{u}^3\right] = \hat{h}\hat{u}\hat{s}\underbrace{\left[\frac{R g_\perp S_c}{\bar{u}_c^2}\right]}_{A} - \frac{5}{9}\hat{h}^2\hat{u}\underbrace{\left[\frac{\left(\frac{\mu_w}{L}\right) g_\perp h_c R}{\bar{u}_c^2}\right]}_{B} - \frac{35}{9}\frac{\hat{u}^2}{\hat{h}^{1/2}}\underbrace{\left[\frac{R\,\chi\Delta\, g_\perp^{1/2}}{h_c^{3/2}\,\bar{u}_c}\right]}_{C}$$

$$A = \frac{R g_\perp S_c}{\bar{u}_c^2} = \frac{R g_\perp \left(\frac{L}{\mu_w}\right)^{-3/4} R^{1/4} (\chi\Delta)^{1/2}}{\left[\left(\frac{L}{\mu_w}\right)^{-3/8} (g_\perp)^{\frac{1}{2}} R^{5/8} (\chi\Delta)^{1/4}\right]^2}$$

$$= \frac{R^{5/4} g_\perp \left(\frac{L}{\mu_w}\right)^{-3/4} (\chi\Delta)^{\frac{1}{2}}}{\left(\frac{L}{\mu_w}\right)^{-3/4} g_\perp R^{5/4} (\chi\Delta)^{1/2}} = 1$$

$$B = \left(\frac{\mu_w}{L}\right) \frac{g_\perp h_c R}{\bar{u}_c^2}$$

$$= \frac{\left(\frac{\mu_w}{L}\right) g_\perp R}{1} \frac{R^{\frac{1}{4}} (\chi D)^{\frac{1}{2}} \left(\frac{L}{\mu_w}\right)^{\frac{1}{4}}}{\left[\left(\frac{L}{\mu_w}\right)^{-3/8} (g_\perp)^{\frac{1}{2}} R^{5/8} (\chi D)^{1/4}\right]^2}$$

$$= \frac{\left(\frac{L}{\mu_w}\right)^{-3/4} g_\perp R^{5/4} (\chi D)^{\frac{1}{2}}}{\left(\frac{L}{\mu_w}\right)^{-3/4} g_\perp R^{5/4} (\chi D)^{1/2}} = 1$$

$$c = \frac{R\,\chi D\, g_\perp^{1/2}}{h_c^{3/2}\,\bar{u}_c}$$

$$= \frac{R\,(\chi D)\,g_\perp^{\frac{1}{2}}}{\left[\left(\frac{L}{\mu_w}\right)^{1/4} R^{1/4} (\chi D)^{1/2}\right]^{3/2} \left(\frac{L}{\mu_w}\right)^{-3/8} (g_\perp)^{\frac{1}{2}} R^{5/8} (\chi D)^{1/4}}$$

$$= \frac{R^{3/8} (\chi D)^{3/4}}{\left(\frac{L}{\mu_w}\right)^{-3/8} \left[\left(\frac{L}{\mu_w}\right)^{3/8} R^{3/8} (\chi D)^{3/4}\right]} = 1$$

$\therefore$ non-dimensional form of (30) in intermediate regime

$$\boxed{\frac{d}{d\hat{x}}\left[K\hat{h}\,\bar{\hat{u}}^3\right] = \hat{h}\,\bar{\hat{u}}\,\hat{S} - \frac{5}{9}\hat{h}^2\bar{\hat{u}} - \frac{35}{9}\frac{\bar{\hat{u}}^2}{\hat{h}^{1/2}}}$$

* Ideally the governing ODEs should ALL contain $\hat{\omega}$
* To enforce the $\hat{\omega}$ dependence we assume a power law scaling between to the characteristic variables for the asyntotic regimes:

(1): $R_a(\hat{\omega} \to 0)$

(2): $R_b(\hat{\omega} \to \infty)$

Now lets non-dimensionalise equ. (29) in R_a regime $(\hat{\omega} \to 0)$

$$dx = R\,d\hat{x} \; ; \; h = h_c \hat{h} \hat{\omega}^{2/7} \quad \bar{u} = \bar{u}_c \hat{\bar{u}} \hat{\omega}^{5/7} ; \quad S = -\frac{dZ}{dx} = S_c \hat{S} \hat{\omega}^{2/7}$$

$\therefore$

$$\frac{d}{R\,d\hat{x}}\left[\gamma \hat{\omega}^{2/7} h_c \hat{h} \left(\hat{\omega}^{5/7} \bar{u}_c \hat{\bar{u}}\right)^2\right] = g_\perp \left(\hat{\omega}^{2/7} h_c \hat{h}\right)\left(\hat{\omega}^{2/7} S_c \hat{S}\right) - \left(\frac{\mu_w}{L}\right) g_\perp \left(\hat{\omega}^{2/7} h_c \hat{h}\right)^2$$

$$\Rightarrow \frac{d}{R\,d\hat{x}}\left[\gamma \hat{h} \hat{\bar{u}}^2\right]\left(\hat{\omega}^{12/7} \cancel{h_c} \bar{u}_c^2\right) = \left(\hat{h}\hat{S}\right)\left[g_\perp \hat{\omega}^{4/7} \cancel{h_c} S_c\right] - \hat{h}^2 \left[\left(\frac{\mu_w}{L}\right) g_\perp \hat{\omega}^{4/7} h_c^{\cancel{2}\,1}\right]$$

$\times$ by $\dfrac{R}{\hat{\omega}^{4/7} \bar{u}_c^2}$

$$\Rightarrow \frac{d}{d\hat{x}}\left[\gamma \hat{h} \hat{\bar{u}}^2\right] \hat{\omega}^{8/7} = \hat{h}\hat{S} \overbrace{\left[\frac{g_\perp S_c R}{\bar{u}_c^2}\right]}^{A} - \hat{h}^2 \overbrace{\left[\frac{\left(\frac{\mu_w}{L}\right) g_\perp h_c R}{\bar{u}_c^2}\right]}^{B} \rightarrow \boxed{\Delta}$$

$$A = \frac{g_\perp S_c R}{\overline{u}_c^{\,2}} = \frac{(g_\perp R)}{1} \frac{\left(\frac{L}{\mu_w}\right)^{-3/4} R^{1/4} (XD)^{1/2}}{\left[\left(\frac{L}{\mu_w}\right)^{-3/8} (g_\perp)^{\frac{1}{2}} R^{5/8} (XD)^{1/4}\right]^2}$$

$$= \frac{g_\perp R \left(\frac{L}{\mu_w}\right)^{-3/4} R^{\frac{1}{4}} (XD)^{\frac{1}{2}}}{\left(\frac{L}{\mu_w}\right)^{-3/4} g_\perp R^{5/4} (XD)^{1/2}} = \underline{1}$$

Similarly:

$$B = \left[\frac{\left(\frac{\mu_w}{L}\right) g_\perp h_c R}{\overline{u}_c^2}\right] = \frac{\left(\frac{\mu_w}{L}\right) g_\perp R}{1} \frac{\left(\frac{L}{\mu_w}\right)^{1/4} R^{1/4} (\chi\Delta)^{1/2}}{\left[\left(\frac{L}{\mu_w}\right)^{-3/8} (g_\perp)^{\frac{1}{2}} R^{5/8} (\chi\Delta)^{1/4}\right]^2}$$

$$= \frac{\left(\frac{L}{\mu_w}\right)^{-3/4} g_\perp R^{5/4} (\chi\Delta)^{\frac{1}{2}}}{\left(\frac{L}{\mu_w}\right)^{-3/4} g_\perp R^{5/4} (\chi\Delta)^{1/2}} = 1$$

$\therefore$ $\boxed{\Delta}$ reduces to:

$$\boxed{\hat{\omega}^{8/7} \frac{d}{d\hat{x}}\left[\chi \hat{h}_a \hat{\overline{u}}_a^2\right] = \hat{h}_a \hat{S}_a - \hat{h}_a^2}$$

Now lets non-dimensionalise equ. (30) in R_a regime ($\hat{\omega} \to 0$)

$$dx = R\,d\hat{x}\;;\quad h = h_c \hat{h}\hat{\omega}^{2/7} \quad \bar{u} = \bar{u}_c \hat{\bar{u}}\hat{\omega}^{5/7};\quad S = -\frac{dZ}{dx} = S_c \hat{S}\hat{\omega}^{2/7}$$

∴ equ. (30) becomes:

$$\frac{d}{R\,d\hat{x}}\left[K h_c \hat{h}\hat{\omega}^{2/7} \bar{u}_c^3 \hat{\bar{u}}^3 \hat{\omega}^{15/7}\right] = g_\perp h_c \hat{h}\hat{\omega}^{2/7} \bar{u}_c \hat{\bar{u}}\hat{\omega}^{5/7} S_c \hat{S}\hat{\omega}^{2/7} - \frac{5}{9}\left(\frac{\mu_w}{L}\right) g_\perp h_c^2 \hat{h}^2 \hat{\omega}^{4/7} \bar{u}_c \hat{\bar{u}}\hat{\omega}^{5/7} - \frac{\frac{35}{9}\chi D g_\perp^{\frac{1}{2}} \bar{u}_c^2 \hat{\bar{u}}^2 \hat{\omega}^{10/7}}{h_c^{\frac{1}{2}}\hat{h}^{1/2}\hat{\omega}^{1/7}}$$

$$\frac{d}{d\hat{x}}\left(K\hat{h}\hat{\bar{u}}^3\right)\left(\frac{h_c \bar{u}_c^3}{R}\hat{\omega}^{17/7}\right) = \hat{h}\hat{\bar{u}}\hat{S}\left(g_\perp h_c \bar{u}_c S_c\right)\hat{\omega}^{9/7} - \frac{5}{9}\hat{h}^2\hat{\bar{u}}\left\{\left(\frac{\mu_w}{L}\right) g_\perp h_c^2 \bar{u}_c\right\}\hat{\omega}^{9/7} - \frac{35}{9}\frac{\hat{\bar{u}}^2}{\hat{h}^{1/2}}\left[\frac{\chi D g_\perp^{\frac{1}{2}} \bar{u}_c^2}{h_c^{1/2}}\right]\hat{\omega}^{9/7}$$

$\times$ by $\dfrac{R}{h_c \bar{u}_c^3 \hat{\omega}^{9/7}}$

$$\Rightarrow \frac{d}{d\hat{x}}\left(K\hat{h}\hat{\bar{u}}^3\right)\hat{\omega}^{8/7} = \hat{h}\hat{\bar{u}}\hat{S}\left[\frac{R g_\perp \cancel{h_c} \cancel{\bar{u}_c} S_c}{\cancel{h_c}\, \bar{u}_c^{\cancel{3}\,2}}\right] - \frac{5}{9}\hat{h}^2\hat{\bar{u}}\left\{\left(\frac{\mu_w}{L}\right)\frac{g_\perp h_c^{\cancel{2}\,1} \cancel{\bar{u}_c} R}{\cancel{h_c}\, \bar{u}_c^{\cancel{3}\,2}}\right\} - \frac{35}{9}\frac{\hat{\bar{u}}^2}{\hat{h}^{1/2}}\left[\frac{R \chi D g_\perp^{\frac{1}{2}} \bar{u}_c^{\cancel{2}}}{\cancel{h_c^{1/2}}\, h_c^{3/2}\, \bar{u}_c^{\cancel{3}\,1}}\right]$$

$$\frac{d}{d\hat{x}}\left[K\hat{h}\hat{u}^{3}\right] = \hat{h}\hat{u}\hat{S}\underbrace{\left[\frac{R g_{\perp} S_c}{\bar{u}_c^{2}}\right]}_{A} - \frac{5}{9}\hat{h}^{2}\hat{u}\underbrace{\left[\frac{\left(\frac{\mu_w}{L}\right) g_{\perp} h_c R}{\bar{u}_c^{2}}\right]}_{B} - \frac{35}{9}\frac{\hat{u}^{2}}{\hat{h}^{1/2}}\underbrace{\left[\frac{R \chi D g_{\perp}^{1/2}}{h_c^{3/2}\bar{u}_c}\right]}_{C}$$

$$A = \frac{R g_{\perp} S_c}{\bar{u}_c^{2}} = \frac{R g_{\perp}\left(\frac{L}{\mu_w}\right)^{-3/4} R^{1/4} (\chi D)^{1/2}}{\left[\left(\frac{L}{\mu_w}\right)^{-3/8}(g_{\perp})^{\frac{1}{2}} R^{5/8} (\chi D)^{1/4}\right]^{2}}$$

$$= \frac{R^{5/4} g_{\perp}\left(\frac{L}{\mu_w}\right)^{-3/4}(\chi D)^{\frac{1}{2}}}{\left(\frac{L}{\mu_w}\right)^{-3/4} g_{\perp} R^{5/4} (\chi D)^{1/2}} = 1$$

$$B = \left(\frac{\mu_w}{L}\right) \frac{g_\perp h_c R}{\bar{u}_c^2}$$

$$= \frac{\left(\frac{\mu_w}{L}\right) g_\perp R}{1} \frac{R^{\frac{1}{4}} (\chi D)^{\frac{1}{2}} \left(\frac{L}{\mu_w}\right)^{\frac{1}{4}}}{\left[\left(\frac{L}{\mu_w}\right)^{-3/8} (g_\perp)^{\frac{1}{2}} R^{5/8} (\chi D)^{1/4}\right]^2}$$

$$= \frac{\left(\frac{L}{\mu_w}\right)^{-3/4} g_\perp R^{5/4} (\chi D)^{\frac{1}{2}}}{\left(\frac{L}{\mu_w}\right)^{-3/4} g_\perp R^{5/4} (\chi D)^{1/2}} = \underline{1}$$

$$c = \frac{R\,\chi D\, g_\perp^{1/2}}{h_c^{3/2}\,\overline{u}_c}$$

$$= \frac{R(\chi D) g_\perp^{\frac{1}{2}}}{\left[\left(\frac{L}{\mu_w}\right)^{1/4} R^{1/4} (\chi D)^{1/2}\right]^{3/2} \left(\frac{L}{\mu_w}\right)^{-3/8} (g_\perp)^{\frac{1}{2}} R^{5/8} (\chi D)^{1/4}}$$

$$= \frac{R^{3/8} (\chi D)^{3/4}}{\left(\frac{L}{\mu_w}\right)^{-3/8} \left[\left(\frac{L}{\mu_w}\right)^{3/8} R^{3/8} (\chi D)^{3/4}\right]} = \underline{1}$$

∴ non-dimensional form of (30) in R_a regime ($\hat{\omega} \to 0$)

$$\hat{\omega}^{8/7} \frac{d}{d\hat{x}}\left[K \hat{h}_a \hat{\overline{U}}_a^3\right] = \hat{h}_a \hat{\overline{U}}_a \hat{S}_a - \frac{5}{9} \hat{h}_a^2 \hat{\overline{U}}_a - \frac{35}{9} \frac{\hat{\overline{U}}_a^2}{\hat{h}_a^{1/2}}$$

Now lets non-dimensionalise equ. (29) in the R_b regime $(\hat{\omega} \to \infty)$

$$dx = R\,d\hat{x} \;;\; h = h_c \hat{h} \hat{\omega}^{\frac{2}{3}} \quad \bar{u} = \bar{u}_c \hat{\bar{u}} \hat{\omega}^{\frac{1}{3}} \quad S = -\frac{dZ}{dx} = S_c \hat{S} \hat{\omega}^{2/3}$$

$$\therefore \quad \frac{d}{R\,d\hat{x}}\left[\gamma \hat{\omega}^{2/3} h_c \hat{h} \left(\hat{\omega}^{\frac{1}{3}} \bar{u}_c \hat{\bar{u}}\right)^2\right] = g_\perp \left(\hat{\omega}^{2/3} h_c \hat{h}\right)\left(\hat{\omega}^{2/3} S_c \hat{S}\right) - \left(\frac{\mu_w}{L}\right) g_\perp \left(\hat{\omega}^{2/3} h_c \hat{h}\right)^2$$

$$\Rightarrow \frac{d}{R\,d\hat{x}}\left[\gamma \hat{h} \hat{\bar{u}}^2\right]\left(\hat{\omega}^{\frac{4}{3}} \cancel{h_c} \bar{u}_c^2\right) = \left(\hat{h}\hat{S}\right)\left[g_\perp \hat{\omega}^{\frac{4}{3}} \cancel{h_c} S_c\right] - \hat{h}^2 \left[\left(\frac{\mu_w}{L}\right) g_\perp \hat{\omega}^{\frac{4}{3}} h_c^{1}\right]$$

$\times$ by $\dfrac{R}{\hat{\omega}^{4/3} \bar{u}_c^2}$

$$\Rightarrow \frac{d}{d\hat{x}}\left[\gamma \hat{h} \hat{\bar{u}}^2\right] = \hat{h}\hat{S} \overbrace{\left[\frac{g_\perp S_c R}{\bar{u}_c^2}\right]}^{A} - \hat{h}^2 \overbrace{\left[\frac{\left(\frac{\mu_w}{L}\right) g_\perp h_c R}{\bar{u}_c^2}\right]}^{B} \rightarrow \boxed{\Delta}$$

$$A = \frac{g_\perp S_c R}{\bar{u}_c^{\,2}} = \frac{(g_\perp R)}{1} \frac{\left(\frac{L}{\mu_w}\right)^{-3/4} R^{1/4} (\chi D)^{1/2}}{\left[\left(\frac{L}{\mu_w}\right)^{-3/8} (g_\perp)^{\frac{1}{2}} R^{5/8} (\chi D)^{1/4}\right]^2}$$

$$= \frac{g_\perp R \left(\frac{L}{\mu_w}\right)^{-3/4} R^{\frac{1}{4}} (\chi D)^{\frac{1}{2}}}{\left(\frac{L}{\mu_w}\right)^{-3/4} g_\perp R^{5/4} (\chi D)^{1/2}} = \underline{1}$$

Similarly :

$$B = \left[\frac{\left(\frac{\mu_w}{L}\right) g_\perp h_c R}{\bar{u}_c^2}\right] = \frac{\left(\frac{\mu_w}{L}\right) g_\perp R}{1} \frac{\left(\frac{L}{\mu_w}\right)^{1/4} R^{1/4} (\chi \Delta)^{1/2}}{\left[\left(\frac{L}{\mu_w}\right)^{-3/8} (g_\perp)^{\frac{1}{2}} R^{5/8} (\chi \Delta)^{1/4}\right]^2}$$

$$= \frac{\left(\frac{L}{\mu_w}\right)^{-3/4} g_\perp R^{5/4} (\chi \Delta)^{\frac{1}{2}}}{\left(\frac{L}{\mu_w}\right)^{-3/4} g_\perp R^{5/4} (\chi \Delta)^{1/2}} = 1$$

$\therefore$ $\boxed{\Delta}$ reduces to :

$$\boxed{\frac{d}{d\hat{x}}\left[\chi \hat{h}_b \hat{\bar{u}}_b^2\right] = \hat{h}_b \hat{S}_b - \hat{h}_b^2}$$

Now lets non-dimensionalise equ. 30 in the R_b regime

$$dx = R\,d\hat{x}\ ;\quad h = h_c \hat{h}\hat{\omega}^{\frac{2}{3}} \quad \bar{u} = \bar{u}_c \hat{\bar{u}}\hat{\omega}^{\frac{1}{3}} \quad S = -\frac{dZ}{dx} = S_c \hat{S}\hat{\omega}^{2/3}$$

∴ equ. 30 becomes:

$$\frac{d}{R\,d\hat{x}}\left[K h_c \hat{h}\hat{\omega}^{2/3} \bar{u}_c^3 \hat{\bar{u}}^3 \hat{\omega}^{3/3}\right] = g_\perp h_c \hat{h}\hat{\omega}^{\frac{1}{3}} \bar{u}_c \hat{\bar{u}}\hat{\omega}^{2/3} S_c \hat{S}\hat{\omega}^{2/3} - \frac{5}{9}\left(\frac{\mu_w}{L}\right) g_\perp h_c^2 \hat{h}^2 \hat{\omega}^{4/3} \bar{u}_c \hat{\omega}^{1/3} \hat{\bar{u}} - \frac{\frac{35}{9}\chi D g_\perp^{\frac{1}{2}}\ \bar{u}_c^2 \hat{\bar{u}}^2 \hat{\omega}^{2/3}}{h_c^{\frac{1}{2}} \hat{h}^{1/2} \hat{\omega}^{1/3}}$$

$$\frac{d}{d\hat{x}}\left(K\hat{h}\hat{\bar{u}}^3\right)\hat{\omega}^{5/3}\left(\frac{h_c \bar{u}_c^3}{R}\right) = \hat{h}\hat{\bar{u}}\hat{S}\hat{\omega}^{5/3}\left(g_\perp h_c \bar{u}_c S_c\right) - \frac{5}{9}\hat{h}^2\hat{\bar{u}}\,\omega^{5/3}\left\{\left(\frac{\mu_w}{L}\right) g_\perp h_c^2 \bar{u}_c\right\} - \frac{35}{9}\frac{\hat{\bar{u}}^2}{\hat{h}^{1/2}}\left[\frac{\chi D g_\perp^{\frac{1}{2}}\bar{u}_c^2}{h_c^{1/2}}\right]\hat{\omega}^{\frac{1}{3}}$$

$\times$ by $\dfrac{R}{h_c \bar{u}_c^3 \hat{\omega}^{5/3}}$

$$\frac{d}{d\hat{x}}\left(K\hat{h}\hat{\bar{u}}^3\right) = \hat{h}\hat{\bar{u}}\hat{S}\left[\frac{R g_\perp \cancel{h_c} \cancel{\bar{u}_c} S_c}{\cancel{h_c}\,\bar{u}_c^{\cancel{3}\,2}}\right] - \frac{5}{9}\hat{h}^2\hat{\bar{u}}\left\{\left(\frac{\mu_w}{L}\right)\frac{g_\perp h_c^{\cancel{2}\,1} \cancel{\bar{u}_c} R}{\cancel{h_c}\,\bar{u}_c^{\cancel{3}\,2}}\right\} - \frac{35}{9}\frac{\hat{\bar{u}}^2}{\hat{h}^{1/2}}\left[\frac{R\chi D g_\perp^{\frac{1}{2}}\bar{u}_c^{\cancel{2}}}{\cancel{h_c^{1/2}}\,h_c^{3/2}\,\bar{u}_c^{\cancel{3}\,1}}\right]\hat{\omega}^{-4/3}$$

$$\therefore \frac{d}{d\hat{x}}\left[K\hat{h}\hat{\bar{u}}^3\right] = \hat{h}\hat{\bar{u}}\hat{S}\underbrace{\left[\frac{R g_\perp S_c}{\bar{u}_c^2}\right]}_{A} - \frac{5}{9}\hat{h}^2\hat{\bar{u}}\underbrace{\left[\frac{\left(\frac{\mu_w}{L}\right) g_\perp h_c R}{\bar{u}_c^2}\right]}_{B} - \frac{35}{9}\frac{\hat{\bar{u}}^2}{\underset{\hat{\omega}^{-4/3}}{\hat{h}^{1/2}}}\underbrace{\left[\frac{R\chi D g_\perp^{1/2}}{h_c^{3/2}\bar{u}_c}\right]}_{C}$$

$$A = \frac{R g_\perp S_c}{\bar{u}_c^2} = \frac{R g_\perp \left(\frac{L}{\mu_w}\right)^{-3/4} R^{1/4} (\chi D)^{1/2}}{\left[\left(\frac{L}{\mu_w}\right)^{-3/8} (g_\perp)^{\frac{1}{2}} R^{5/8} (\chi D)^{1/4}\right]^2}$$

$$= \frac{R^{5/4} g_\perp \left(\frac{L}{\mu_w}\right)^{-3/4} (\chi D)^{\frac{1}{2}}}{\left(\frac{L}{\mu_w}\right)^{-3/4} g_\perp R^{5/4} (\chi D)^{1/2}} = 1$$

$$B = \left(\frac{\mu_w}{L}\right) \frac{g_\perp h_c R}{\bar{u}_c^2}$$

$$= \frac{\left(\frac{\mu_w}{L}\right) g_\perp R}{1} \frac{R^{\frac{1}{4}} (\chi D)^{\frac{1}{2}} \left(\frac{L}{\mu_w}\right)^{\frac{1}{4}}}{\left[\left(\frac{L}{\mu_w}\right)^{-3/8} (g_\perp)^{\frac{1}{2}} R^{5/8} (\chi D)^{1/4}\right]^2}$$

$$= \frac{\left(\frac{L}{\mu_w}\right)^{-3/4} g_\perp R^{5/4} (\chi D)^{\frac{1}{2}}}{\left(\frac{L}{\mu_w}\right)^{-3/4} g_\perp R^{5/4} (\chi D)^{1/2}} = \underline{1}$$

$$c = \frac{R\,\chi D\, g_\perp^{1/2}}{h_c^{3/2}\,\bar{u}_c}$$

$$= \frac{R(\chi D)g_\perp^{\frac{1}{2}}\ \hat{\omega}^{-4/3}}{\left[\left(\frac{L}{\mu_w}\right)^{1/4} R^{1/4} (\chi D)^{1/2}\right]^{3/2} \left(\frac{L}{\mu_w}\right)^{-3/8} (g_\perp)^{\frac{1}{2}} R^{5/8} (\chi D)^{1/4}}$$

$$= \frac{R^{3/8}(\chi D)^{3/4}\ \hat{\omega}^{-4/3}}{\left(\frac{L}{\mu_w}\right)^{-3/8}\left[\left(\frac{L}{\mu_w}\right)^{3/8} R^{3/8} (\chi D)^{3/4}\right]}$$

$$= \omega^{-4/3}$$

$\therefore$ non-dimensional form of (30) in R_b regime is :

$$\boxed{\frac{d}{d\hat{x}}\left[K \hat{h}_b \hat{\bar{u}}_b^{3}\right] = \hat{h}\,\hat{\bar{u}}_b \hat{S}_b - \frac{5}{9}\hat{h}_b^{2}\hat{\bar{u}}_b - \frac{35}{9}\,\frac{\hat{\bar{u}}_b^{2}}{\hat{h}_b^{1/2}}\,\omega^{-4/3}}$$

stream function $\psi(x,z)$

$$\frac{\partial \psi}{\partial z} = u(x,z)$$

But $u\left(\hat{\eta}(x,z)\right) = \bar{u}(x,z) f\left(\hat{\eta}\right)$

$$\partial z = -\partial \hat{\eta}$$

$$\therefore d\psi = \bar{u} f \, d\hat{\eta}$$

$$\int_0 d\psi$$