Energy Dynamics

Energy Dynamics

Energy Balance (Equations 36 & 37 in Assignment)

$$\frac{\partial}{\partial t}\left[\left(\frac{77}{96}\right)h\bar{u}^2\right] + \frac{\partial}{\partial x}\left[K h \bar{u}^3\right] = -g_\perp h\bar{u}\left(\frac{\partial Z}{\partial x}\right) - \frac{5}{9}\frac{\mu_w}{L} g_\perp h^2 \bar{u} - \frac{35}{9}\chi\Delta\sqrt{\frac{g_\perp}{h}}\,\bar{u}^2 \rightarrow (1)$$

rate of change of K.E

Divergence of K.E Flux

Work of normal stress gradient

Energy dissipat^n by Wall shear stresses (Wall friction)

Energy dissipation by internal shear stresses

$$\frac{N}{m^2}\times\frac{1}{\cancel{m}}\times\frac{\cancel{m}}{s}$$

$$\frac{N}{m^2 . s}$$

$$+\rho u\frac{\partial u}{\partial t} + \rho\underbrace{\left(u^2\frac{\partial u}{\partial x} + uw\frac{\partial u}{\partial z}\right)}_{= u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}} = -\cancel{\rho}\,\frac{u}{\cancel{\rho}}\frac{\partial \sigma}{\partial x} - \cancel{\rho}\,2u\frac{\tau_w}{\cancel{\rho}L} + \cancel{\rho}\,\frac{u}{\cancel{\rho}}\frac{\partial \tau}{\partial z} + u g_{\parallel} \rightarrow (2)$$

Comments

* Equ. (1) is the depth integrated form; eq (2) is the local values @ each (x,z) locat^n
* All terms have dimensions of energy flux $[J/m^2 . s]$
* Equ. (2) also has a term for gravitational production $[u g \sin\beta]$; however this term cancels out in derivation of (1) when we choose $\alpha = \beta$
* Hung chose to study terms 2 & 5 of equat^n (2)

Kinetic Energy Flux

Let $K' = \frac{1}{2} M v^2$ be the usual kinetic energy of mass M moving with velocity $\vec{V}$

Define $K \equiv \frac{K'}{M} = \frac{1}{2} V^2$ as the kinetic energy per unit mass

By definition, the kinetic energy efflux, q_K, is given by:

$q_K \equiv K \rho \vec{V} \cdot d\vec{A}$ with corresponding dimensions: $[q_K] = \frac{J}{kg} \frac{kg}{m^3} \frac{m}{s} m^2 = \frac{J}{s}$

The corresponding kinetic energy flux, Q_K, is defined as:

$Q_K \equiv K \rho \vec{V}$ with dimensions:

$$[Q_K] = \frac{J}{kg} \frac{kg}{m^3} \frac{m}{s} = \frac{J}{m^2 \cdot s}$$

Our task is to show that $\vec{\nabla} \cdot (K \rho \vec{V})$ is:

* equivalent to term 2 of equation 2, to within scalar constants

Consider $\vec{\nabla}\cdot(K\rho\vec{\nabla}) \equiv \frac{\partial}{\partial x_j}\left[\frac{\rho}{2}|V|^2 V_j\right]$ ... Einstein notation & summation convention

where $j=1,2 \ni (x_1,x_2)=(x,z)$; $(V_1,V_2)=(u,w)$ & $|V|=\sqrt{V_iV_i}=\sqrt{u^2+w^2}$

$\Rightarrow |V| = (u^2+w^2)$

Therefore:

$$\frac{\partial}{\partial x_j}\left[\frac{\rho}{2}|V|^2 V_j\right] = \frac{\partial}{\partial x}\left[\frac{\rho}{2}(u^2+w^2)u\right] + \frac{\partial}{\partial z}\left[\frac{\rho}{2}(u^2+w^2)w\right]$$

$$= \frac{\rho}{2}\left[\frac{\partial}{\partial x}\left(u^3+uw^2\right)\right] + \frac{\rho}{2}\left[\frac{\partial}{\partial z}\left(u^2w+w^3\right)\right]$$

$$= \frac{\rho 3u^2}{2}\frac{\partial u}{\partial x} + \frac{\rho w^2}{2}\frac{\partial u}{\partial x} + \frac{\rho u}{2}\left[2w\frac{\partial w}{\partial x}\right] + \frac{\rho}{2}w2u\frac{\partial u}{\partial z} + \frac{\rho}{2}3w^2\frac{\partial w}{\partial z}$$

$$= \frac{\rho}{2}\frac{\partial u}{\partial x}\left[3u^2+w^2\right] + \frac{\rho}{2}2uw\cancelto{0}{\frac{\partial w}{\partial x}} + \rho uw\frac{\partial u}{\partial z} + \frac{\rho 3w^2}{2}\cancelto{0}{\frac{\partial w}{\partial z}}$$ ... by shallow flow approx

$$= \frac{\rho}{2}\frac{\partial u}{\partial x}\left[3u^2+\cancelto{0}{w^2}\right] + \rho uw\frac{\partial u}{\partial z}.$$

But $\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \Rightarrow w^2\frac{\partial u}{\partial u}+w^2\frac{\partial w}{\partial z}=0 \Rightarrow w^2\frac{\partial u}{\partial x} = -w^2\cancelto{0}{\frac{\partial w}{\partial z}}$ ... by shallow flow approximation

$$\therefore \boxed{\frac{\partial}{\partial x_j}\left[\frac{\rho}{2}|V|^2 V_j\right] = \rho\left(\frac{3}{2}\right)u^2\frac{\partial u}{\partial x} + \rho uw\frac{\partial u}{\partial z}} \rightarrow 4$$

Now consider the dimensions of term 2 in equation (2)

$$\left[\text{Term } 2\right] = \frac{m^2}{s^2}\,\frac{m}{s}\,\frac{1}{m}\,\frac{kg}{m^3} = \Big[\underbrace{\underbrace{\left(\frac{kg\cdot m}{s^2}\right)}_{N} \times m}_{J}\Big] \times \frac{1}{m^2 \times s}$$

$$\therefore \left[\text{Term } 2\right] = \frac{J}{m^2\cdot s} \equiv \left[\text{Kinetic Energy Flux}\right]$$

Similarly, the dimensions of $\vec{\nabla}\cdot(K\rho\vec{v})$ can be gleaned from equation (4):

$$\left[\vec{\nabla}\cdot(K\rho\vec{v})\right] = \frac{J}{m^2\cdot s}$$

Therefore:

* Term 2 is dimensionally equivalent to the divergence of the kinetic energy flux
* $\vec{\nabla}\cdot(K\rho\vec{v})$ also brings in the correct factor of $\left(\frac{3}{2}\right)$ to the 1st term of equ. (4)
* We will drop the density term in numerical calcs. as it simply serves to scale the results by a constant ρ in our case of incompressible flow

Noting that $u = \bar{u} F(\hat{\eta}) = \bar{u}\left[\frac{7}{3} - \frac{35}{6}\hat{\eta}^{3/2} + \frac{7}{2}\hat{\eta}^{5/2}\right]$

$\therefore \dfrac{\partial u}{\partial x} = F(\hat{\eta})\left(\dfrac{\partial \bar{u}}{\partial x}\right)$

Soln. of $\dfrac{\partial \bar{u}}{\partial x}$:

* Solve for $u(x)$
* Fit spline to u vs x
* Numerically differentiate spline

Energy dissipation by granular viscous stress: Term 5 of Equ. 2

$u\frac{\partial \tau}{\partial z}$: Consider $\frac{\partial}{\partial z}(u\tau)$

$$\frac{\partial}{\partial z}(u\tau) = u\frac{\partial \tau}{\partial z} + \tau\frac{\partial u}{\partial z}$$

Integrate over depth, $h(x)$:

$$\Rightarrow \int_{\underset{\sim}{z}}^{\tilde{z}} \frac{\partial}{\partial z}(u\tau)\,dz = \int_{\underset{\sim}{z}}^{\tilde{z}} u\frac{\partial \tau}{\partial z}\,dz + \int_{\underset{\sim}{z}}^{\tilde{z}} \tau\frac{\partial u}{\partial z}\,dz \rightarrow 5$$

from boundary conditions

$$\text{But } \int_{\underset{\sim}{z}}^{\tilde{z}} \frac{\partial}{\partial z}(u\tau)\,dz = \int_{\underset{\sim}{z}}^{\tilde{z}} d(u\tau) = \left[u\tau\right]_{\underset{\sim}{z}}^{\tilde{z}} = \underbrace{\tilde{u}\cancelto{0}{\tilde{\tau}} - \cancelto{0}{\underset{\sim}{u}}\,\underset{\sim}{\tau}} = 0$$

So 5 reduces to: $$\boxed{\int_{\underset{\sim}{z}}^{\tilde{z}} u\frac{\partial \tau}{\partial z}\,dz = -\int_{\underset{\sim}{z}}^{\tilde{z}} \tau\frac{\partial u}{\partial z}\,dz}$$ $\rightarrow$ This is equivalent to: $\int u\,d\tau = -\int \tau\,du$

$\div$ by $dz \Rightarrow$ $\int u\frac{d\tau}{dz} = -\int \tau\frac{du}{dz}$

Finally, equating the integrands $\Rightarrow$ $$\boxed{u\frac{d\tau}{dz} = -\tau\frac{du}{dz} = -\tau\dot{\gamma}}$$

So to summarise: $u\frac{\partial \tau}{\partial z} = -\tau\frac{\partial u}{\partial z}$ & this is equivalent to

$$u\frac{d\tau}{dz} = -\tau\dot{\gamma} \quad \rightarrow (6)$$

Now lets expand $\left[-\tau\frac{\partial u}{\partial z}\right]$ using the linearised $\mu(I)$-rheology

$$\tau = \mu_0\sigma + \chi\Delta\dot{\gamma}\sqrt{\rho\sigma}$$

Noting that in the α-tilted frame, all terms involving (α) can be ignored

$\Rightarrow$ the shear stress (τ) reduces to:

$$\tau = \chi\Delta\dot{\gamma}\sqrt{\rho\sigma}$$

$$\therefore u\frac{\partial \tau}{\partial z} = -\tau\left(\frac{\partial u}{\partial z}\right) = -\chi\Delta\dot{\gamma}\sqrt{\rho\sigma}\,\frac{\partial u}{\partial z}$$

$$= -\chi\Delta\dot{\gamma}\sqrt{\rho\sigma}\,\dot{\gamma} \quad \text{... from } (6)$$

$$= -\chi\Delta\dot{\gamma}^2\left[\rho\left\{\rho g\cos\beta(\tilde{z}-z)\right\}\right]^{\frac{1}{2}} = -\chi\Delta\dot{\gamma}^2\rho\left[g_\perp\hat{\eta}h\right]^{\frac{1}{2}} = \varphi_v$$

In Einstein notation:

$$\varphi_v = -\chi\Delta|\dot{\gamma}|^2\rho\left[g_\perp\hat{\eta}h\right]^{\frac{1}{2}}, \text{ where } \dot{\gamma}_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \ni \dot{\gamma} = \frac{\partial u}{\partial z} + \cancelto{0}{\frac{\partial w}{\partial x}} = \frac{\partial u}{\partial z}$$

... by shallow flow approx.

Derivation of shear rate: Method 1

As shown in Q6: $u(x,z) = \bar{u}(x)\, f\big(\hat{\eta}(x,z)\big)$,

where $f(\hat{\eta}) = \frac{7}{3} - \frac{35}{6}\hat{\eta}^{3/2} + \frac{7}{2}\hat{\eta}^{5/2}$

$$\therefore\ \dot{\gamma} = \frac{\partial u}{\partial z} = \bar{u}(x)\,\frac{\partial f(\hat{\eta})}{\partial z}$$

Noting that $\partial z = -h\,\partial\hat{\eta}$

$$\Rightarrow\ \dot{\gamma} = -\frac{\bar{u}(x)}{h(x)}\,\frac{\partial f}{\partial \hat{\eta}}$$

$$= -\frac{\bar{u}}{h}\left[-\frac{\overset{18}{\cancel{36}}}{{}_{2}\cancel{6}}\,\frac{\overset{1}{\cancel{3}}}{\cancel{2}_{1}}\,\hat{\eta}^{\frac{1}{2}} + \frac{7}{2}\,\frac{5}{2}\,\hat{\eta}^{3/2}\right]$$

$$= \frac{\bar{u}}{h}\left[9\hat{\eta}^{\frac{1}{2}} - \frac{35}{4}\hat{\eta}^{3/2}\right]$$

$$\therefore\ \dot{\gamma}(x,z) = \frac{\bar{u}(x)}{h(x)}\left[9\hat{\eta}^{\frac{1}{2}} - \frac{35}{4}\hat{\eta}^{3/2}\right] \quad \text{where} \quad 0 \leqslant \hat{\eta} \leqslant 1$$

$$\dot{x} = \frac{\sqrt{g\cos\beta}}{\chi D}\left[\eta^{\frac{1}{2}}\,\frac{h\,\mu_w}{L} - \frac{\mu_w}{L}\,\eta^{3/2}\right]$$ ... from (17) & (18) in assignment

$$= \frac{\mu_w\sqrt{g\cos\beta}}{\chi L D}\left[\eta^{\frac{1}{2}}h - \eta^{3/2}\right]$$

$$= \left(\frac{4}{35}h^{5/2}\right)\frac{\mu_w\sqrt{g\cos\beta}}{\chi L D}\left[\frac{35}{4h^{5/2}}\left(\eta^{\frac{1}{2}}h - \eta^{3/2}\right)\right]$$

$$= \bar{u}\left[\frac{35}{4h^{5/2}}\left(\eta^{\frac{1}{2}}h - \eta^{3/2}\right)\right]$$

$$\dot{X} = \bar{u}\left[\frac{35}{4h^{5/2}}\left(\eta^{\frac{1}{2}}h - \eta^{3/2}\right)\right]$$

$$= \frac{35\bar{u}}{4}\left[\eta^{\frac{1}{2}}h^{+3/2} - \eta^{3/2}h^{-5/2}\right]$$

$$= \frac{35\bar{w}}{4}\left[\hat{\eta}^{\frac{1}{2}}h^{\frac{1}{2}}h^{-3/2} - \hat{\eta}^{3/2}h^{3/2}h^{-5/2}\right]$$

$$= \frac{35\bar{w}}{4h}\left[\hat{\eta}^{\frac{1}{2}} - \hat{\eta}^{3/2}\right]$$

Steady State Energy Dynamics for experimental data

Ignoring the shallow flow approximation $\Rightarrow \frac{\partial w}{\partial x} \neq 0$ & $\frac{\partial w}{\partial z} \neq 0$

Cauchy Stress: $\sigma = -p\mathbb{1} + \tau$ ... $-p\underline{I} = \sigma_{hyd}$ ≡ hydrostatic stress

where p ≡ isotropic pressure & τ ≡ deviatoric stress

In the x,z direction $\Rightarrow$

$$\begin{bmatrix} -p & \tau_{xz} \\ \tau_{zx} & -p \end{bmatrix} = \underbrace{-p\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{-\underbrace{\begin{bmatrix}\text{Hydrostatic} \\ \text{Stress}\end{bmatrix}}_{\text{isotropic stress}}} + \underbrace{\begin{bmatrix} 0 & \tau_{xz} \\ \tau_{zx} & 0 \end{bmatrix}}_{\text{Deviatoric Stress}}$$

Also: $\sigma = \sigma^T \Rightarrow \sigma$ is symmetric

$\tau = \sigma - \frac{1}{2}tr(\sigma)\mathbb{1}$, where $\mathbb{1}$ ≡ unit tensor (in 2D)

$$\frac{1}{2}tr(\sigma)\mathbb{1} = \frac{1}{2}(-2p)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -p & 0 \\ 0 & -p \end{bmatrix} = -p\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -p\mathbb{1}$$

By the linearised $\mu(I)$ constitutive law

$$\tau = (\mu_0 + \chi I)\frac{D}{\|D\|}p \text{, where } L = \nabla V = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial z} \end{bmatrix} \text{ ... velocity gradient with } V = u + w$$

$$D = \frac{1}{2}\left(L + L^T\right) \text{... symmetric strain rate tensor}$$

$$= \frac{1}{2}\left(\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial z} \end{bmatrix} + \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} & 2\frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} D_{xx} & D_{xz} \\ D_{zx} & D_{zz} \end{bmatrix}$$

$$\|D\| = \sqrt{\frac{1}{2}(\mathrm{tr}D)^2} = \sqrt{\frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)^2} = \frac{1}{\sqrt{2}}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)$$

$$\sqrt{\frac{1}{2}\mathrm{tr}(D^2)} = \sqrt{\frac{1}{2}\left[4\left(\frac{\partial u}{\partial x}\right)^2 + 4\left(\frac{\partial w}{\partial z}\right)^2\right]} = \sqrt{2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]}$$

$$I = \frac{2\|D\|d}{\sqrt{p/\rho}} = \frac{\frac{2d}{\sqrt{2}}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)}{\sqrt{p/\rho}} = \frac{\sqrt{2}\,d}{\sqrt{p/\rho}}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)$$

$$= \sqrt{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]^{\frac{1}{2}} = \sqrt{2}\left\{\left[\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right]^2 - 2\frac{\partial u}{\partial x}\frac{\partial w}{\partial z}\right\}^{\frac{1}{2}}$$

<u>NB</u>: In using the linearised $\mu(I)$-rheology τ_{xx} & τ_{zz} are non-zero $\because$ D_{xx} & D_{zz} are non-zero

By the linearised $\mu(I)$ constitutive law

$$\tau = (\mu_0 + \chi I)\frac{D}{\|D\|}P \text{, where } L = \nabla V = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial z} \end{bmatrix} \text{ ... velocity gradient with } V = u + w$$

$$D = \frac{1}{2}\left(L + L^T\right) \text{ ... symmetric strain rate}$$

$$= \frac{1}{2}\left(\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial z} \end{bmatrix} + \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} & 2\frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} D_{xx} & D_{xz} \\ D_{zx} & D_{zz} \end{bmatrix}$$

$$D^2 = \frac{1}{4}\begin{bmatrix} 4\left(\frac{\partial u}{\partial x}\right)^2 & \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 \\ \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 & 4\left(\frac{\partial w}{\partial z}\right)^2 \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial u}{\partial x}\right)^2 & \frac{1}{4}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{4}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 & \left(\frac{\partial w}{\partial z}\right)^2 \end{bmatrix}$$

$$\|D\| = \sqrt{\frac{tr(D^2)}{2}} = \frac{1}{\sqrt{2}}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]^{\frac{1}{2}}$$

$$I = \frac{2\|D\|d}{\sqrt{P/\rho_p}} = \frac{\sqrt{2}\,d\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]^{\frac{1}{2}}}{\sqrt{P/\rho_p}} = \frac{\sqrt{2}\,d}{\sqrt{P/\rho_p}}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]^{\frac{1}{2}}$$

NB: In using the viscoplastic rheology, τ_{xx} & τ_{zz} are non-zero $\because$ D_{xx} & D_{zz} are non-zero

$$\because \quad \tau = \begin{bmatrix} \tau_{xx} & \tau_{xz} \\ \tau_{zx} & \tau_{zz} \end{bmatrix} \quad \ni \quad \tau = \tau^T \quad \because \quad \sigma = \sigma^T$$

$\Longrightarrow$ τ is symmetric

$$= \frac{p(\mu_0 + \chi I)}{\|D\|} \begin{bmatrix} D_{xx} & D_{xz} \\ D_{zx} & D_{zz} \end{bmatrix}$$

$$= \frac{p(\mu_0 \quad \chi I)}{\|D\|} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{\partial w}{\partial z} \end{bmatrix}$$

Differential Momentum Equations

If the stresses @ the walls of the differential volume are related to the stresses @ the center via a 1st order Taylor series expansion, then the following properties arise for the wall stresses:

(i) Normal stresses σ_{ii} are directed away from plane & pressure p is directed into the plane $\ni$ $\sigma_{ii} = -p$ for

(ii) Stresses at $x_i + \frac{\Delta x_i}{2}$ are directed in the positive x_i-direction

(iii) Stresses at $x_i - \frac{\Delta x_i}{2}$ are directed in the negative x_i-direction

NB: * These properties ensure a self-consistent set of equations for the surface stresses.
* The external forces must still be correctly accounted for i.t.o. magnitude & direction

Using these properties for our granular flow problem along with the:
* Constitutive law and
* Relevant external forces

We may write out the general x & z momentum balances x-momentum balance @ steady-state:

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = g_{11} - \frac{2\tau_w}{\rho L} + \frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial z} + \frac{1}{\rho}\frac{\partial \tau_{xx}}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

side wall

$\tau_{xx} = -p$

$$\Rightarrow \boxed{u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = g_{11} - \frac{2\tau_w}{\rho L} + \frac{1}{\rho}\frac{\partial \tau_{xx}}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial z}} \rightarrow (1)$$

NB: The x-component of the shear stress tensor is: $\tau_{xz} + \tau_{xx}$

Also, τ_{xx} acts on differential area $dA_1 = (dz)L \ni$ our force due to τ_{xx} is: $(dz)L\left[\tau_{xx}\big|_{x+dx} - \tau_{xx}\big|_{x}\right]$

$\therefore$ when we $\div$ by $\rho(dx)(dz)L$, the dz term cancels.

So in the limit $dx \to 0$ & $dz \to 0$, we get: $\left[\frac{\partial \tau_{xx}}{\partial x}\right]$

z - momentum balance @ steady-state

$\tau_{xx} = -p$

$$w\frac{\partial w}{\partial z} + u\frac{\partial w}{\partial x} = +g\cos\beta - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{zz}}{\partial z}$$

$\therefore$ z-momentum balance becomes

$$\boxed{w\frac{\partial w}{\partial z} + u\frac{\partial w}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial z} = -g_{\perp} + \frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{zz}}{\partial z}} \longrightarrow 2$$

We have shown earlier, that

$$\Rightarrow \frac{u}{\rho}\frac{\partial \tau_{xz}}{\partial z} = -\frac{\tau_{xz}}{\rho}\frac{\partial u}{\partial z}; \text{ similarly: } \quad \frac{u}{\rho}\frac{\partial \tau_{xx}}{\partial x} = -\frac{\tau_{xx}}{\rho}\frac{\partial u}{\partial x};$$

$$\& \ \frac{w}{\rho}\frac{\partial \tau_{xz}}{\partial x} = -\frac{\tau_{xz}}{\rho}\frac{\partial w}{\partial x}; \quad \frac{w}{\rho}\frac{\partial \tau_{zz}}{\partial z} = -\frac{\tau_{zz}}{\rho}\frac{\partial w}{\partial z}$$

$\therefore$ 1 becomes:

$$\boxed{u\left(\frac{\partial u}{\partial x}\right) + w\left(\frac{\partial u}{\partial z}\right) + \frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right) = g_{\parallel} - \frac{2\tau_w}{\rho L} - \frac{\tau_{xx}}{\rho}\left(\frac{\partial u}{\partial x}\right) - \frac{\tau_{xz}}{\rho}\left(\frac{\partial u}{\partial z}\right)} \rightarrow 1*$$

$\therefore$ 2 becomes:

$$\boxed{w\left(\frac{\partial w}{\partial z}\right) + u\left(\frac{\partial w}{\partial x}\right) + \frac{1}{\rho}\left(\frac{\partial p}{\partial z}\right) = -g_{\perp} - \frac{\tau_{xz}}{\rho}\left(\frac{\partial u}{\partial z}\right) - \frac{\tau_{zz}}{\rho}\left(\frac{\partial w}{\partial z}\right)} \rightarrow 2*$$

* We now scale 1* & 2*, ∋ their terms represent dimensions of Energy flux, by doing following:

1* $\times u\rho$

2* $\times w\rho$

$$\text{(1*)} \times u\rho \Rightarrow \rho u\left[u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x}\right] = \rho u\left[g_{\parallel} - \frac{2\tau_w}{\rho L} + \frac{1}{\rho}\frac{\partial \tau_{xx}}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial z}\right]$$

$$\text{(2*)} \times w\rho \Rightarrow \rho w\left[u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z}\right] = \rho w\left[-g_{\perp} + \frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{zz}}{\partial z}\right]$$

Adding the above equat^{ns}:

$$\Rightarrow u\rho\left[u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x}\right] + \rho w\left[u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z}\right]$$

$$= u\rho\left[g_{\parallel} + \frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial z} + \frac{1}{\rho}\frac{\partial \tau_{xx}}{\partial x} - \frac{2\tau_w}{\rho L}\right] + \rho w\left[-g_{\perp} + \frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{zz}}{\partial z}\right]$$

$$= \rho u g_{\parallel} + \underbrace{\cancel{\rho}\frac{u}{\cancel{\rho}}\frac{\partial \tau_{xz}}{\partial z}}_{-\tau_{xz}\frac{\partial u}{\partial z}} + \underbrace{\cancel{\rho}\frac{u}{\cancel{\rho}}\frac{\partial \tau_{xx}}{\partial x}}_{-\tau_{xx}\frac{\partial u}{\partial x}} - \frac{\cancel{\rho}2u\tau_w}{\cancel{\rho}L} - w g_{\perp}\rho + \underbrace{\cancel{\rho}\frac{w}{\cancel{\rho}}\frac{\partial \tau_{xz}}{\partial x}}_{-\tau_{xz}\frac{\partial w}{\partial x} + f(\mu_0)} + \underbrace{\cancel{\rho}\frac{w}{\cancel{\rho}}\frac{\partial \tau_{zz}}{\partial z}}_{-\tau_{zz}\frac{\partial w}{\partial z} + f(\mu_0)} \rightarrow \text{(3)}$$

NB: $f(\mu_0)$ is not shear rate dependent & will be excluded in our final energy dissipat^n analysis.

So 3 becomes:

$$\rho u\left[u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial x}\right]+\rho w\left[u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial z}\right]=$$

$$\rho\left[g_{\parallel}-\frac{\tau_{xz}}{\rho}\frac{\partial u}{\partial z}-\frac{\tau_{xx}}{\rho}\frac{\partial u}{\partial x}-\frac{2\tau_w}{\rho L}\right]+\rho\left[-g_{\perp}-\frac{\tau_{xz}}{\rho}\frac{\partial w}{\partial x}-\frac{\tau_{zz}}{\rho}\frac{\partial w}{\partial z}\right]+f(\mu_0)$$

Isolating the shear dissipation terms:

$$\varphi_v' = -\left[\underbrace{\tau_{xx}\left(\frac{\partial u}{\partial x}\right)}_{(i)}+\underbrace{\tau_{xz}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)}_{(ii)}+\underbrace{\tau_{zz}\left(\frac{\partial w}{\partial z}\right)}_{(iii)}\right] \rightarrow 4$$

Energy dissipation rate by internal shear stresses

(i) $= \tau_{xx} \frac{\partial u}{\partial x}$

$$= \left[\left(\mu_0 + \chi I \right) \frac{D_{xx}\, p}{\|D\|} \right] \frac{\partial u}{\partial x}$$

$$= \left[\left(\mu_0 + \frac{\chi\, 2\|D\|\, D}{\sqrt{p/\rho}} \right) \frac{\frac{\partial u}{\partial x}\, p}{\|D\|} \right] \frac{\partial u}{\partial x} \qquad \therefore D_{xx} = \frac{\partial u}{\partial x}$$

$$= \left[\frac{\mu_0 p}{\|D\|} + \frac{2\chi D p}{\sqrt{p/\rho}} \right] \left(\frac{\partial u}{\partial x} \right)^2$$

$$(ii) = \tau_{xz}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$= \left[\left(\mu_0 + \chi I\right)\frac{D_{xz}p}{\|D\|}\right]\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$= \frac{1}{2}\left(\frac{\mu_0 p}{\|D\|} + \frac{2\chi D\, p}{\sqrt{p/\rho}}\right)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 \qquad \ldots\; D_{xz} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$(iii)\ \tau_{zz}\frac{\partial w}{\partial z} = \left[\left(\mu_0 + \chi I\right)\frac{D_{zz}p}{\|D\|}\right]\frac{\partial w}{\partial z}$$

$$= \left[\frac{\mu_0 p}{\|D\|} + \frac{2\chi D\, p}{\sqrt{p/\rho}}\right]\left(\frac{\partial w}{\partial z}\right)^2 \qquad \ldots\; D_{zz} = \frac{\partial w}{\partial z}$$

$$\therefore \text{(i)} + \text{(ii)} + \text{(iii)} = \left[\frac{\mu_0 p}{\|D\|} + \frac{2\chi D p}{\sqrt{P/\rho}}\right]\left\{\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2\right\}$$

consider :

$$\underbrace{\left(\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x}\right)^2}_{\text{=0 by the incompressibility condition}} = \left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial w}{\partial z}\frac{\partial u}{\partial x}$$

$$\Longrightarrow \left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 = -2\frac{\partial w}{\partial z}\frac{\partial u}{\partial x}$$

$$\therefore \text{(i)} + \text{(ii)} + \text{(iii)} = \left[\underbrace{\frac{\mu_0 p}{\|D\|}}_{(a)} + \underbrace{\frac{2\chi D p}{\sqrt{P/\rho}}}_{(b)}\right]\left[-2\frac{\partial w}{\partial z}\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2\right]$$

(b): Rate Dependent dissipation

(a) : Dissipation required to :
- * Prop up bed to repose angle α
- * Overcome yield criterion of bed

Focusing on rate-dependent dissipation, ie, excluding terms in μ_0, equ. 4 becomes

$$\Rightarrow \Phi_v' = \chi D \sqrt{p\rho}\left[4\frac{\partial w}{\partial z}\frac{\partial u}{\partial x} - \left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2\right] \rightarrow (5)$$

$\therefore$ The energy dissipation rate (per unit density) by internal shear stresses is:

$$\Phi_v = \frac{\Phi_v'}{\rho} = \chi D \sqrt{\frac{p}{\rho}}\left[4\frac{\partial w}{\partial z}\frac{\partial u}{\partial x} - \left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2\right]$$

NB: When considering the total energy dynamics, the Φ_v contribution in our energy balance equation must be taken to the L.H.S. of equation

$$\Longrightarrow \Phi_{total} \equiv \Phi = \Phi_k - \Phi_v \text{ yields the total energy dynamics}$$

Consider $\vec{\nabla}\cdot(K\rho\vec{V}) \equiv \frac{\partial}{\partial x_j}\left[\frac{\rho}{2}|V|^2 V_j\right]$ ... Einstein notation & summation convention

where $j=1,2 \ni (x_1,x_2)=(x,z)$; $(V_1,V_2)=(u,w)$ & $|V| = \sqrt{V_iV_i} = \sqrt{u^2+w^2}$

$\Rightarrow |V| = (u^2+w^2)$

Therefore:

$$\frac{\partial}{\partial x_j}\left[\frac{\rho}{2}|V|^2 V_j\right] = \frac{\partial}{\partial x}\left[\frac{\rho}{2}(u^2+w^2)u\right] + \frac{\partial}{\partial z}\left[\frac{\rho}{2}(u^2+w^2)w\right]$$

$$= \frac{\rho}{2}\left[\frac{\partial}{\partial x}\left(u^3+uw^2\right)\right] + \frac{\rho}{2}\left[\frac{\partial}{\partial z}\left(u^2w+w^3\right)\right]$$

$$= \frac{\rho 3u^2}{2}\frac{\partial u}{\partial x} + \frac{\rho w^2}{2}\frac{\partial u}{\partial x} + \frac{\rho u}{2}\left[2w\frac{\partial w}{\partial x}\right] + \frac{\rho}{2}w2u\frac{\partial u}{\partial z} + \frac{\rho}{2}3w^2\frac{\partial w}{\partial z}$$

$$= \frac{\rho}{2}\frac{\partial u}{\partial x}\left[3u^2+w^2\right] + \frac{\rho}{2}2uw\frac{\partial w}{\partial x} + \rho uw\frac{\partial u}{\partial z} + \frac{\rho 3w^2}{2}\frac{\partial w}{\partial z}$$

$$= \rho\left[\frac{1}{2}\frac{\partial u}{\partial x}\left(3u^2+w^2\right) + uw\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) + \frac{3}{2}\frac{\partial w}{\partial z}w^2\right]$$

$$\therefore \boxed{\vec{\nabla}\cdot(K\rho\vec{V}) = \rho\left[\frac{1}{2}\frac{\partial u}{\partial x}\left(3u^2+w^2\right) + uw\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) + \frac{3}{2}w^2\left(\frac{\partial w}{\partial z}\right)\right]}$$

$$\therefore \quad \boxed{\varphi_k' = \rho\left[\frac{1}{2}\frac{\partial u}{\partial x}\left(3u^2+w^2\right) + uw\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) + \frac{3}{2}w^2\left(\frac{\partial w}{\partial z}\right)\right]}$$

$$\boxed{\varphi_v = \chi D\sqrt{P/\rho}\left[4\frac{\partial w}{\partial z}\frac{\partial u}{\partial x} - \left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2\right]}$$

So finally, the total energy dynamics per unit density is:

$$\varphi = \frac{\varphi_k'}{\rho} - \frac{\varphi_v'}{\rho}$$

$$\therefore \quad \boxed{\varphi = \frac{1}{2}\frac{\partial u}{\partial x}\left(3u^2+w^2\right) + uw\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) + \frac{3}{2}w^2\left(\frac{\partial w}{\partial z}\right) + \chi D\sqrt{\frac{P}{\rho}}\left[\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2 - 4\frac{\partial w}{\partial z}\frac{\partial u}{\partial x}\right]}$$

NB: $\sqrt{\frac{P}{\rho}} = \sqrt{\frac{\rho g h\cos\beta}{\rho}} = \sqrt{(g\cos\beta)h} = \sqrt{g_\perp h} \Rightarrow$ no density dependence

Implications of scaling the Energy Dynamics by Density (ρ_p)

* The density dependence is removed $\Rightarrow$ the energy dynamics of 2 drums with different density particles (everything else the same) can be compared

* In terms of DEM data & coarse graining, the particle density ρ_p is important $\therefore$ it does not cancel out

* For granular mixtures of different densities, we can use a weighted average of the densities

$\therefore$

$$\varphi_k(x,z) = \frac{3}{2}u^2\left(\frac{\partial u}{\partial x}\right) + uw\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + w^2\left[\frac{3}{2}\frac{\partial w}{\partial z} + \frac{1}{2}\frac{\partial u}{\partial x}\right]$$

$$\therefore \vec{\nabla}\cdot\left(K\rho\vec{V}\right) = \rho\left[\frac{1}{2}\frac{\partial u}{\partial x}\left(3u^2+w^2\right) + uw\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + \frac{3}{2}w^2\left(\frac{\partial w}{\partial z}\right)\right]$$