Stream Function Basics

* For an incompressible flow in the $(x\text{-}z)$ plane, the stream function is defined as:

  * $\dfrac{\partial \psi}{\partial z} = u$ & $\dfrac{\partial \psi}{\partial x} = -w$, where (u, w) are the x & z velocity components

* The stream function is only defined wherever the flow is defined; and since no flow occurs for $z(x) > \tilde{z}(x)$, we choose $\psi(x, \tilde{z})$ as our reference and set it to zero.

  $\therefore \psi(x, z) = 0 \quad \ldots \; z(x) \geq \tilde{z}(x)$

  So the ψ-value at any other streamline represents flow between $\tilde{z}(x)$ and that streamline.

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$$u(x,z)=\bar{u}(x)\, f\left(\hat{\eta}(x,z)\right) \quad \ldots \quad \underline{\text{Flowing Layer}}: \; f(\hat{\eta}) = \frac{7}{3} - \frac{35}{6}\hat{\eta}^{3/2} + \frac{7}{2}\hat{\eta}^{5/2}$$

The stream function $\psi(x,z)$, for an incompressible flow, is defined as:

$$\frac{\partial \psi}{\partial z} = u \rightarrow (1) \quad \& \quad \frac{\partial \psi}{\partial x} = -w \rightarrow (2)$$

Integrate (1): $\Rightarrow$

$$\int \frac{\partial \psi}{\partial z}\, dz = \int u\, dz$$

$$\Rightarrow \int d\psi = \bar{u}(x) \int f(\hat{\eta})\, dz$$

$$\Rightarrow \psi = \bar{u} \int f(\hat{\eta})(-h\, d\hat{\eta}) + g(x) \quad \ldots \; dz = -h\, d\hat{\eta}, \quad g(x) \equiv \text{an unknown } f^n$$

$$= -\bar{u}(x) h(x) \int f(\hat{\eta}) \quad + g(x)$$

$$= -\bar{u}(x) h(x) \underbrace{\left[\frac{7}{3}\hat{\eta} - \frac{7}{3}\hat{\eta}^{5/2} + \hat{\eta}^{7/2}\right]}_{F(\hat{\eta}(x,z))} + g(x)$$

$$\therefore \psi(x,z) = -\bar{u}(x) h(x) F(\hat{\eta}) + g(x) \rightarrow (1)$$

$$@\; z = \tilde{z}, \; \hat{\eta} = 0 \Rightarrow F = 0$$

Also, $@\; z = \tilde{z}, \; \psi(x,\tilde{z}) = 0$ ... This is our reference choice

applying B.C. to (1) $\Rightarrow \; 0 = g(x)$

$\therefore$ (1) becomes: $\psi(x,z) = -\bar{u}(x) h(x) F\left(\hat{\eta}(x,z)\right)$

Denoting stream function in flowing layer as ψ_f, we have:

$$\psi_f(x,z) = -\bar{u}(x) h(x) F\left(\hat{\eta}(x,z)\right) \rightarrow (2)$$

with $F\left(\hat{\eta}(x,z)\right) = \frac{7}{3}\hat{\eta} - \frac{7}{3}\hat{\eta}^{5/2} + \hat{\eta}^{7/2}$

Solid body region

For a clockwise rotating drum: $u = \omega z \rightarrow (3)$

$w = -\omega x \rightarrow (4)$

From defⁿ of stream function in solid body region ψ_s for incompressible flow:

$$\frac{\partial \psi_s}{\partial z} \equiv u = \omega z \rightarrow (5) \quad \& \quad \frac{\psi_s}{\partial x} \equiv -w = +\omega x \rightarrow (6)$$

Integrate (5):

$$\Rightarrow \int d\psi_s = \omega \int z\,dz + p(x)$$

unknown f^n

$$\Rightarrow \psi_s = \frac{1}{2}\omega z^2 + p(x)$$

@ $z = \underset{\sim}{z}$, $\hat{\eta} = 1 \Rightarrow F = 1$, $\psi_f = \psi_s$

$$\psi_s(z=\underset{\sim}{z}) = \frac{1}{2}\omega \underset{\sim}{z}^2 + p(x)$$

$$\psi_f(z=\underset{\sim}{z}) = -\bar{u}(x)h(x)(1)$$

$$\therefore \frac{1}{2}\omega \underset{\sim}{z}^2 + p(x) = -\bar{u}(x)h(x)$$

$$\Rightarrow p(x) = -\bar{u}(x)h(x) - \frac{1}{2}\omega \underset{\sim}{z}^2$$

$$\therefore \psi_s(x,z) = \cancel{\frac{1}{2}\omega \underset{\sim}{z}^2} - \bar{u}(x)h(x) - \cancel{\frac{1}{2}\omega \underset{\sim}{z}^2}$$

$$\Rightarrow \psi_s(x,z) = -\bar{u}(x)h(x) \rightarrow (7)$$

Combining ② & ⑦ into a single function, we have :

$$\psi(x,z) = -\overline{u}(x)\,h(x)\,F\big(\hat{\eta}(x,z)\big) \longrightarrow ⑧$$

where

$$F\big(\hat{\eta}(x,z)\big) = \begin{cases} 0 & \ldots \quad \hat{\eta} \leq 0 \\ \frac{7}{3}\hat{\eta}^{3/2} - \frac{7}{3}\hat{\eta}^{5/2} + \hat{\eta}^{7/2} & \ldots \quad 0 \leq \hat{\eta} \leq 1 \\ 1 & \ldots \quad \hat{\eta} \geq 1 \end{cases}$$

Corrections to stream function : $\Delta\psi$

* As it stands, $\psi(x,z)$ does not recover the small velocity components due to solid body rotation alone !
* Consider the ffg. stream f^n associated with solid body rotation alone :

$$\psi_0 = +\frac{1}{2}\omega\left[x^2+z^2\right] \;\therefore\; \underbrace{\frac{\partial\psi_0}{\partial z} \equiv u \Rightarrow u = \omega z}_{\text{correctly yields } u \text{ in solid-body}} \quad \& \quad \underbrace{\frac{\partial\psi_0}{\partial x} \equiv -w \Rightarrow w = -\frac{\partial\psi_0}{\partial x} = -\omega x}_{\text{correctly yields } w \text{ in solid-body}}$$

Consistent with shallow flow approximation, we need the z-components to vanish in flowing layer

$$\Longrightarrow \Delta\psi = \psi_0 - \frac{1}{2}\omega x^2 = \cancel{\frac{1}{2}\omega x^2} + \frac{1}{2}\omega z^2 - \cancel{\frac{1}{2}\omega x^2}$$

So finally : $\Delta\psi = +\frac{1}{2}\omega z^2$ ... Is the required correction that recovers solid body velocity components in the flowing layer consistent with the shallow flow approximation

So the final form of the stream function is:

$$\psi(x,z) = -\bar{u}(x)\,h(x)\,F\big(\hat{\eta}(x,z)\big) + \underbrace{\tfrac{1}{2}\omega z^2}_{\Delta\psi} \rightarrow (9)$$

where

$$F\big(\hat{\eta}(x,z)\big) = \begin{cases} 0 & \ldots \quad \hat{\eta} \leq 0 \\ \frac{7}{3}\hat{\eta}^{3/2} - \frac{7}{3}\hat{\eta}^{5/2} + \hat{\eta}^{7/2} & \ldots \quad 0 \leq \hat{\eta} \leq 1 \\ 1 & \ldots \quad \hat{\eta} \geq 1 \end{cases}$$

and $\Delta\psi = \frac{1}{2}\omega z^2$ is a correction to recover the small x-velocity components (only) due to solid body rotation

NB: We can use properties of $\psi(x,z) \equiv \tilde{\psi}$ to find free surface & basal interface

Free surface from stream function

* @ $z = \tilde{z}$, $\psi(x, \tilde{z}) = 0$

  So finding the free surface $\tilde{z}(x)$ is equivalent to finding the z-values when the stream function is zero

* The corresponding contour line for $\psi(x, z) = 0$ forms a closed loop with $\tilde{z}(x)$ & the drum wall below $\tilde{z}(x)$

Basal Interface from $\psi(x,z)$

At the basal interface $\underset{\sim}{u}(x,z) = 0$

But $\dfrac{\partial \psi}{\partial z} = u$ ... by def^{n}

$\therefore$ $\dfrac{\partial \psi}{\partial z} = 0$ locates the points of $\psi(x,z)$ that lie on basal interface

$$\Rightarrow \quad -\bar{u}h\left(\frac{\partial F}{\partial z}\right) + \omega z = 0$$

Now solve for z

$$\Rightarrow \quad -\bar{u}h\left(\frac{\partial F}{-\partial \hat{\eta}\, h}\right) + \omega z = 0$$

$$\Rightarrow \quad \bar{u}\frac{\partial F}{\partial \hat{\eta}} + \omega z = 0$$

$$\Rightarrow \quad \bar{u}f(\hat{\eta}) + \omega z = 0$$

$$\Rightarrow \quad \bar{u}f\left(\frac{\Sigma - z}{h}\right) + \omega z = 0$$

$$\Rightarrow \quad \bar{u}\left[\frac{7}{3} - \frac{35}{6}\left(\frac{\Sigma - z}{h}\right)^{3/2} + \frac{7}{2}\left(\frac{\Sigma - z}{h}\right)^{5/2}\right] + \omega z = 0$$

Basal interface from transformed stream function

Consider the following transformed stream function $\psi_t(x,z)$:

$$\psi_t(x,z) = \bar{u}(x)\, h(x)\, F\big(\hat{\eta}(x,z)\big) + \frac{1}{2}\,\omega\left[R^2 - x^2\right]$$

On plotting the streamlines for $\psi_t(x,z)$, the following is observed:

(i) The streamlines exist between the numerically determined free surface $\overset{\approx}{z}_n(x)$ & basal surface $\underset{\sim}{z}_n(x)$

(ii) The streamline closest to $\overset{\approx}{z}_n(x)$ is nearly parallel to $\overset{\approx}{z}_n(x)$

(iii) The streamline closest to $\underset{\sim}{z}_n(x)$ is nearly parallel to $\underset{\sim}{z}_n(x)$

(iv) @ $\psi_t = 0$, the streamline lies atop $\underset{\sim}{z}_n(x)$ $\rightarrow$ so no new (improved) basal interface obtained here!!!

Generating contours from stream function

* Consider the range of values $\psi(x_i, z_i)$, where $x_i, z_i \in [-R ; R]$
* The corresponding z values are obtained from equation ⑨ :

$$z_i = \pm \sqrt{\frac{2}{\omega}\left[\psi(x_i, z_i) + \bar{u}(x_i) h(x_i)\right]}$$

NB : This makes sense ∵ an existent contour is a closed loop

$\Rightarrow$ for each x_i, we have 2 values for z_i