Recent developments in models, experiments, and simulation of dynamic flows of grains

Over the past fifteen years there has been a paradigm shift in the continuum modelling of granular materials; most notably with the development of rheological models, such as the $\mu(I)$ rheology. This rheology is based on experimental measurements and Discrete Particle Method/Discrete Element Method (DPM/DEM) simulation data from six different steady state flow configurations, which show that the friction μ is a monotonically increasing function of the nondimensional inertial number I. The inertial number itself is defined as $I = \dot{\gamma}d/(p/\rho_\star)^{1/2}$, where $\dot{\gamma}$ is the shear rate, p is the pressure, $\rho_\star$ is the particle density and d is the particle size. The $\mu(I)$ rheology is therefore a non-trivial generalization of the constant Coulomb friction coefficient, that introduces shear rate, pressure and particle-size dependence into the law. This paper reviews the experimental observations that led to the scalar $\mu(I)$ rheology and shows how it was generalized into tensorial form (Jop, Forterre & Pouliquen 2006 Nature vol. 44, pp. 727–730).

If the granular material is assumed to be incompressible then the $\mu(I)$ rheology has a similar mathematical structure to the Navier-Stokes equations of fluid mechanics. This is a useful analogy, which enables existing numerical methods to be relatively easily adapted to solve problems of practical interest. The challenge lies in the fact that the granular viscosity is not constant, but is linearly dependent on the friction and pressure, and is inversely proportional to the shear rate. This makes the equations much harder to solve. However, early simulations of granular column collapses and silo flow show that the rheology has the potential to solve real world problems using a relatively simple continuum theory. This is an exciting development, as continuum theories do not suffer from the computational restrictions of DPM/DEM in having to resolve every single particle in a large domain.

The original form of the $\mu(I)$ rheology is well-posed for a wide range of inertial numbers, but at high and low inertial numbers it is mathematically ill-posed. This means that the growth rate of linear instabilities is unbounded in the high number wave limit. This is a real problem, because it implies that numerical simulations will not converge to solution as a numerical grid is refined, and wave-like instabilities can grow catastrophically and break the numerical methods. One way of circumventing this is to use the partially regularized $\mu(I)$ rheology (Barker & Gray 2017 J. Fluid Mech. vol. 828, pp. 5–32.), which assumes that there is no yield stress, i.e. $\mu(0) = 0$, and uses the ill-posedness analysis to define a new frictional function $\mu(I)$ that is well-posed for all inertial numbers below a maximum threshold ($I_{\text{max}} \simeq 17$). This theory enables reliable computations to be made with the incompressible $\mu(I)$ rheology that are grid converged over a wide range of inertial numbers. In particular, it is possible to couple the theory to recent models for particle-size segregation in multi-component mixtures (Gray

& Ancey 2011 J. Fluid Mech., vol. 678, pp. 353–588) and solve for segregating flows on chutes and in partially filled rotating drums. This is an important step towards simulating realistic granular materials that are composed of particles of differing sizes, as well as frictional and chemical properties. The resulting model is intimately coupled, because the inertial number is dependent on the particle size, so the local friction changes as the size distribution evolves during the flow. It is this fact that often leads to unexpected segregation in many industrial processes, which degrade the quality of the resulting products, and can cause severe flow problems.

A new class of granular rheologies have also recently been developed, which are called the Compressible I-Dependent Rheologies (CIDR). These essentially combine rate-independent Critical State Soil Mechanics (CSSM) with the rate dependence of the $\mu(I)$ rheology. In steady-state configurations these are designed to reduce to the classical $\mu(I)$ rheology, so they can capture the same steady-state behaviour observed in experiments and DPM/DEM simulations. However, by construction these theories are guaranteed to be mathematically well-posed and thermodynamically consistent. These models differ from the original $\mu(I)$ rheology in their transient response in time-dependent problems. The CIDR model is a framework, rather than a specific model, with considerable flexibility still available to design yield and dilatancy functions to match experiments and discrete particle simulations. Comparison of the inertial CIDR model (Schaeffer et al. 2019 J. Fluid Mech. 2019, vol. 874, pp. 926–951) with DPM/DEM simulations in one-dimensional time-dependent gravitationless shear, shows that it accurately captures the right transient response, whereas the compressible $\mu(I)$ rheology blows up catastrophically. The CIDR models include the important physical effect of compressibility, are mathematically well posed and probably show the greatest potential for the future, but at present a great deal of work needs to done to narrow down the appropriate yield and dilatancy functions, as well as develop robust numerical methods to solve the resulting equations in two and three dimensions.

The $\mu(I)$ rheology has also been incorporated into depth-averaged models. These exploit the shallowness of a flow in a particular direction to integrate through the flow depth and thereby remove one spatial dimension from the problem. Such models are potentially useful for computing flows in industrial transfer chutes, and are highly developed because of their application to large scale natural hazards, such as snow avalanches, rockfalls and debris flows. In particular, the depth-averaged $\mu(I)$ rheology naturally leads to the derivation of a dynamic basal friction law for chute flows that is dependent on the flow depth h and the Froude number $Fr = |\overline{\mathbf{u}}|/\sqrt{gh\cos\zeta}$, where $\overline{\mathbf{u}}$ is the depth-averaged velocity, g is the gravitational acceleration and ζ is the slope inclination angle. It is measurements of this basal friction law (Pouliquen 1999 Phys. Fluids vol. 11, 542–548) which actually led to the original development of the $\mu(I)$ rheology. Indeed, careful measurements of the depth-averaged flow velocity and thickness as a function of slope angle still provides the primary means of determining the parameters in the $\mu(I)$ rheology. Tipping chute experiments should therefore become a routine rheological test to determine these parameters in future.

The $\mu(I)$ rheology is a very useful rheology that does a very good job of simulating many flows, however it is not a universal panacea. There are many flow features that it can not explain. For instance, when a steady uniform flow on a rough inclined plane is brought to rest by cutting off the supply, the grains do not all run off the chute, but they leave a deposit of thickness $h_{\text{stop}}(\zeta)$. To get this material to flow again, the chute must be inclined to a steeper angle $\zeta_{\text{start}}(h)$, which is the inverse function of $h_{\text{start}}(\zeta)$. As a result, there is a metastable range of thicknesses $h \in [h_{\text{stop}}, h_{\text{start}}]$ within which flowing and static states can coexist. This leads to very rich flow behaviour, with the formation of (i) deposition waves, (ii) retrogressive failures, (iii) self-channelizing flows that are bounded by static levees and (iv) the formation of discrete wave pulses that progressively erode and deposit material as they propagate downslope. All of this behaviour can not be predicted by the $\mu(I)$ rheology alone, and are examples of non-local behaviour, i.e. the local behaviour of the flow is affected by motion, or otherwise, of the grains some distance away. In the case of these chute flow phenomena, it is the presence of the static basal boundary that is felt some distance away. Recent non-local models which capture some of these phenomena will be discussed.

Potential chapter titles

1. Experimental observations
2. The $\mu(I)$ rheology for granular flows
3. Simulation methods (Silos, rotating drums and column collapses)
4. Ill-posedness and partial regularization of the $\mu(I)$ rheology
5. Coupling rheology and segregation in granular flows
6. Well-posed Compressible I-dependent rheologies (CIDR)
7. Depth-averaged models incorporating the $\mu(I)$ rheology
8. Beyond the $\mu(I)$ rheology – weakly non-local behaviour